

SPHEROIDAL GROUPS, VIRTUAL COHOMOLOGY AND LOWER DIMENSIONAL G -SPACES

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This paper is dedicated to the memory of my beloved friend and colleague
Sam Gitler.

ABSTRACT. A space is defined to be “ n -spheroidal” if it has the homotopy type of an n -dimensional CW-complex X with $H_n(X; \mathbb{Z})$ not zero and finitely generated. A group G is called “ n -spheroidal” if its classifying space $K(G, 1)$ is n -spheroidal. Examples include fundamental groups of compact manifold $K(G, 1)$ ’s. Moreover, the class of groups G which are n -spheroidal for some n , is closed under products, free products, and group extensions. If Y is a space with $\pi_1(Y)$ n -spheroidal, and if $H_k(Y; \mathbb{F}_p)$ is non-zero and finitely generated, and if $H_i(Y; \mathbb{F}_p) = 0$ for $i > k$, then $H_{n+k}(\bar{Y}; \mathbb{F}_p) \neq 0$ for \bar{Y} a finite sheeted covering space of Y . Hence, $\dim(Y) \geq n + k$. Thus, it follows that if $\dim(Y) < n$, and if $H_k(Y; \mathbb{F}_p) \neq 0$ and $H_i(Y; \mathbb{F}_p) = 0$ for $i > k > 0$, then $H_k(Y; \mathbb{F}_p)$ is not finitely generated. Similar results follow for $Y \subset K(G, 1)$.

1. INTRODUCTION

We call an n -dimensional CW complex X “ n -spheroidal” if the integral homology $H_n(X)$ is (a) non trivial and (b) finitely generated, (“weakly n -spheroidal” without (b)). Note that this implies that $H_n(X)$ is free abelian, being isomorphic to the cycles in $C_n(X)$, and this holds for any finite sheeted covering of X . Also every homology class in $H_n(X)$ is detected by a map of X into S^n , which is the origin of this choice of notation. Similar statements hold for chain complexes, and for groups by applying this condition to the classifying space of G .

Let p be prime and \mathbb{F}_p denote the field with p elements.

Theorem 1.1. *Let X be a CW complex with weakly n -spheroidal fundamental group and with universal covering space Y such that $H_k(Y; \mathbb{F}_p)$ is non zero and finitely generated*

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for some prime p , and that $H_q(Y; \mathbb{F}_p) = 0$ for $q > k$. Then the dimension of Y is greater than or equal to $n + k$. (For example if an n -spheroidal group G acts freely on $M \times R^k$, where M is a compact manifold, then k is greater than or equal to n).

This implies that for a free G CW complex Y , G n -spheroidal and $\dim Y < n$, the top homology with \mathbb{F}_p coefficients of Y is not finitely generated for any prime p .

I thank the referee for comments, and particularly for calling my attention to the two papers of N. Petrosyan which contain related results [3, Proposition 1.3] and [4, Theorem 4.2] which are proved in similar way. I thank J. Kollár for drawing my attention to such problems (see Section 3).

2. SPHEROIDAL GROUPS

For any group G one can construct a CW complex $K(G, 1)$ (called the classifying space of G) with fundamental group G and all other homotopy groups trivial, (see for example [[1]). The homotopy type of $K(G, 1)$ depends only on G , and its cohomology is called the cohomology of the group. (An algebraic definition using projective resolutions is also standard.)

We call a group G n -spheroidal if $K(G, 1)$ is n -spheroidal up to homotopy (similarly for weakly n -spheroidal).

Proposition 2.1. *Let $f : G \rightarrow L$ be a homomorphism of groups where L is finite and let D be the kernel of f , so that D has finite index in G . If G is n -spheroidal then so is D (similarly for weakly n -spheroidal). The converse holds if $K(G, 1)$ is n -dimensional.*

Proof. We use the spectral sequence for

$$K(D, 1) \longrightarrow K(G, 1) \longrightarrow K(L, 1).$$

To verify that $H_n(D)$ is non-zero, and is finitely generated if $H_n(G)$ is, we use the fact that L is finite, so that the reduced homology $\overline{H}_*(K(L, 1))$ is a \mathbb{Z} -torsion module, (annihilated by $|L|$). It is clear that $K(D, 1)$ is n -dimensional since it is the quotient of the (n -dimensional) universal space for G by the subgroup D . \square

The converse follows easily with the extra condition.

A CW complex X is simply called “spheroidal” if it is n -spheroidal for some n . If $K(G, 1)$ has the homotopy type of a closed oriented n -manifold, it is n -spheroidal,

and finitely generated free groups are 1-spherical. The class of spherical groups is closed under finite direct products and finite free products, which shows that there are many such groups. It will follow from our arguments below, that extensions of (weakly) n -spherical groups by m -spherical groups are (weakly) $(n+m)$ -spherical, and extensions of n -spherical groups by m -spherical groups are $(n+m)$ -spherical.

Thus the class of (weakly) n -spherical groups is large.

3. THE RESULTS

Theorem 3.1. *Let the group G be weakly n -spherical, and let Y be a projective $\mathbb{Z}G$ -chain complex such that $Y_k = 0$ for $k > n$ (for example, the chains of a G -free CW complex of dimension $\leq n$). If $j > 0$ is maximal such that $H_j(Y; \mathbb{F}_p)$ is non-zero, and if $H_j(Y; \mathbb{F}_p)$ is finitely generated, then $H_{n+j}(Y/D; \mathbb{F}_p)$ is not zero for some subgroup of finite index D of G .*

Corollary 3.2. *If G is weakly n -spherical and acts freely on Y of dimension $< n$, then the top non-zero dimensional $H_j(Y; \mathbb{F}_p)$ is not finitely generated for every prime p .*

Question: With the hypothesis of Corollary 3.2, can one conclude that for every $m > 0$, $H_m(Y)$ non-zero implies $H_m(Y)$ is not finitely generated?

Theorem 3.3. *Let X be an n -spherical space, $f : E \rightarrow X$ be a fibre space with fibre Y . If for some prime p , $H_i(Y; \mathbb{F}_p)$ is finitely generated and non zero, and $H_k(Y; \mathbb{F}_p)$ is zero for $k > i$, then $H_{n+i}(E'; \mathbb{F}_p)$ is non zero for some pullback E' of E over X' a finite sheeted covering of X , corresponding to some subgroup D of finite index in $G = \pi_1(X)$.*

Proof. The action of G on the fibre Y induces an action of G on $H_*(Y; \mathbb{F}_p)$, in other words, a homomorphism of G to $\text{Aut}(H_i(Y; \mathbb{F}_p))$. The latter is a finite group because it is a linear group of a finite dimensional vector space over \mathbb{F}_p . If D is the kernel of this homomorphism, then D acts as the identity on $H_i(Y; \mathbb{F}_p)$. Note that $H_n(X)$ is a free abelian group for an n dimensional CW complex X . Consider the pullback $E' \rightarrow X'$, where X' is the covering of X corresponding to the subgroup D of G . Then the highest dimensional group in E^2 of the spectral sequence for $E' \rightarrow X'$ is

$$E_{n,i}^2 = H_n(X'; H_i(Y; \mathbb{F}_p)) = H_n(X') \otimes H_i(Y; \mathbb{F}_p)$$

which is non zero since $H_n(X')$ is free and the action of the fundamental group on the homology of the fibre is trivial. It is in the kernel of every differential since $H_s(Y; \mathbb{F}_p) = 0$ for $s > i$, and is never a boundary since $H_t(X; \mathbb{F}_p) = 0$ for $t > n$. (“The upper right hand corner” argument). Hence $E_{n,i}^2 = E_{n,i}^\infty$, and $H_{n+i}(E'; \mathbb{F}_p)$ is therefore non zero. \square

Theorem 3.1 follows by setting $X = K(G, 1)$ etc.

Corollary 3.4. *Let G be an n -spherical group. If Y is a G subset of the universal G space (i.e., the universal cover of $K(G, 1)$) and if the highest dimensional non-zero cohomology group $H^k(Y)$ is finitely generated, then $k = 0$ and Y is acyclic so that $H^*(Y/G)$ is isomorphic to $H^*(K(G, 1))$.*

Corollary 3.5. *With the hypotheses of Corollary 3.4, suppose in addition $K(G, 1)$ is minimal (for example a compact n -manifold), then Y is the universal G space.*

Proof. If Y/G is a subspace of a compact manifold $K(G, 1)$ with the same homology, then $Y/G = K(G, 1)$ and Y is the universal G space. Any subset of a compact manifold, having the same top dimensional homology, must be the whole manifold. \square

Corollary 3.5 answers in the affirmative the following question put to me by J. Kollár which stimulated this study, [2]:

Kollár’s Question: Let $f: \mathbb{R}^n \rightarrow (S^1)^n$ be the universal cover. Suppose X is a subset of $(S^1)^n$ such that $f^{-1}(X) \subseteq \mathbb{R}^n$ has the homotopy type of a finite complex; is X equal to \mathbb{R}^n ?

REFERENCES

- [1] K. Brown, *Cohomology of groups*, Springer Verlag.
- [2] János Kollár and John Pardon, *Algebraic varieties with semialgebraic universal cover*, J. Topol. **5**, (2012), no. 1, 199–212.
- [3] N. Petrosyan, *Jumps in cohomology and free group actions*, Journal of Pure and Applied Algebra, **210**, (2007), 695–703.
- [4] N. Petrosyan, *Action-induced nested classes of groups and jump cohomology*, Forum Mathematicum, **27**, no. 3, (2015) 1591–1612.

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